ALGORITHMS: Through the Ages and Around the World

Joyce F. Fischer
Jean F. Davis

Webster's Collegiate Dictionary defines an algorithm as "a rule of procedure for solving a mathematical problem that frequently involves repetition of an operation." (p. 22) The word “algorithm” comes from the name of the 8th century Arab mathematician Mohammed al-Khowarizmi. He wrote two books, one about arithmetic and one about algebra, that explained the Hindu-Arabic numerals and methods of calculation. Readers of the Latin translations began to attribute the system to him and to call the new methods of calculation “algorismi.” Over time the word became “algorithm,” and came to mean an orderly, repetitive scheme. (Long, p. 186)

The phrase “alternative algorithm” refers to procedures different from those currently taught in most public school classrooms in the United States. Standard or traditional algorithms provide a first look at the four basic operations. As the numbers become very large or more difficult, alternative algorithms provide an interesting and generally, logically diverse, method of approach for students who are having trouble learning the basic school algorithms. Many alternative algorithms are found in the public domain and additional methods are being devised each day. Some references to Mathematics Education texts that contain various alternative algorithms are cited at the end of this paper.

Here are some reasons why teachers should teach and use alternative algorithms in their classrooms:

1. Alternative algorithms accommodate different learning styles.
2. Alternative algorithms demonstrate that there IS more than just one way to solve a problem.
3. Alternative algorithms are fun!
4. Alternative algorithms provide a means so that we can appreciate the efforts of other people in other times and places.

Addition

Adding by Place Values

This procedure uses a "left-to-right" approach (Bennett, p. 141), which was used by the early Hindus and later by the Europeans. Add the number in the highest place value first, then each lower place value until you reach the ones place value.

124	}	100 + 300	+	20 + 90	+	4 + 2
+ 392		400	+	110	+	6
				516		

Across, Down, and Across

Here is a different addition algorithm, which uses the expanded form of a number and the distributive law to simplify computation. Write each number in expanded form horizontally first. Then add the numbers in the appropriate place values (vertically). Then add these numbers horizontally.

124	=	1 x 100	+ 2 x 10	+ 4 x 1	
+ 392	=	3 x 100	+ 9 x 10	+ 2 x 1	
		4 x 100	+ 11 x 10	+ 6 x 1	
		400	+ 110	+ 6	= 516

Count and Strike: Base Ten

This algorithm gives us a way to keep track of groups of the base without having to remember multi-digit numbers. Starting at the top right place value (the ones), add down until you get one group of the base or more. Make a strike and write the remainder small and to the right of the number that was struck out. At the very bottom of the column write the last remainder. Count the strikes and put that number at the top of the next place value. Always start counting at the top. Keep going until there are no more place values. This method works well for non-standard bases.

Base Ten Problem

1124
1321
1109
204
+ 392

Base Ten Problem rewritten with more space for clarity

	1	1	2	
	1	1	2	4
	1	3	2	1
	1	1	0	~~9~~4
		2	0	4
+		~~3~~1	~~9~~5	~~2~~0
	4	1	5	0

Count and Strike: Other Bases

This algorithm also translates easily to addition and multiplication (repeated addition) in other bases. For other bases, the number in the group (the base) is the only thing that changes.

Base Five with more space for clarity.

	1	1	1	2	
		1	1	1	4
		2	1	1	~~3~~2
		~~1~~0	1	~~2~~1	1
+		1	~~1~~0	1	~~2~~0
	1	1	0	2	0

Base Eight with more space for clarity.

	1	2		1	2	
		2	3	1	5	4
		2	~~7~~2	1	~~2~~1	3
		1	~~6~~0	1	4	~~5~~4
+		~~3~~2	1	2	1	~~7~~3
	1	2	1	6	6	3

Subtraction

Equal Addition

The equal addition method is taught in schools in Australia and some Latin American countries. It is also sometimes taught in the northeastern part of the United States.

Add the same number to the minuend and the subtrahend, but in a different form, then subtract. This method never involves borrowing.

Original Problem with space		New Problem with space
3 1	Add ten ones	3 ^{1}1
		3
- 2 9	Add one ten	- ~~2~~ 9
		2
2 0 7	Add ten tens	2 ^{1}0 7
		2
- 1 4 2	Add one hundred	- ~~1~~ 4 2
		6 5

If more than one place value needs to be adjusted, do all of them at once, then subtract.

2 1 1	Add ten ones, Add ten tens	2 ^{1}1 ^{1}1
		1 3
- 2 3	Add one ten, Add one hundred	- ~~2~~ 3
		1 8 8

Austrian Method for Subtraction

The Austrian method is really the same idea as Equal Addition, but written in a different horizontally oriented form.

Consider the problem 764 – 348.

$$
\begin{aligned}
764 - 348 &= (700 + 60 + 4) - (300 + 40 + 8) \\
&= (700 + 60 + \underbrace{10 + 4}) - (300 + \underbrace{40 + 10} + 8) \quad \longleftarrow \text{ Add 10} \\
&= (700 + 60 + 14) - (300 + 50 + 8) \\
&= (700 - 300) + (60 - 50) + (14 - 8) \\
&= 400 + 10 + 6 \\
&= 416
\end{aligned}
$$

In the second step, we added 10 to each number. In the third step, the 10 was combined with the 4 ones in the minuend and with the 4 tens in the subtrahend, making the subtraction possible.

Here is another example: 3002 – 1875

$$
\begin{aligned}
&3002 - 1875 \\
&= (3000 + 2) - (1000 + 800 + 70 + 5) \\
&= (3000 + 1000 + 100 + 10 + 2) - (1000 + 1000 + 800 + 100 + 70 + 10 + 5) \\
&= (3000 + 1000 + 100 + 12) - (2000 + 900 + 80 + 5) \\
&= (3000 - 2000) + (1000 - 900) + (100 - 80) + (12 - 5) \\
&= 1000 + 100 + 20 + 7 \\
&= 1127
\end{aligned}
$$

Method of Complements for Subtraction

This method for subtraction is used in Europe. With this algorithm, we start by adding some power of the base in the middle of the problem then we subtract it out at the end of the problem. Beginning with the middle number, subtract the digit below this number and then add the digit above this number. If you need to carry, add the carried number to the top number. The leftmost digit of the answer (difference) will always be a one. Cross it out and you have the difference. Here are several examples.

Original Problem two digits	New Problem with more space	
	^{1}	
5 3	5 3	ones column 10 – 7 + 3
<u>- 2 7</u>	9 ^{1}0	tens column 9 – 2 + 5
	<u>- 2 7</u>	hundreds column bring down
	~~1~~ 2 6	the carried one

Cross out the one in the leftmost column. The final answer is 26.

Original Problem four digits	New Problem with more space	
	^{1}	
3 0 0 2	3 0 0 2	ones column 10 – 5 + 2
<u>- 1 8 7 5</u>	9 9 9 ^{1}0	tens column 9 – 7 + 0
	<u>- 1 8 7 5</u>	hundreds column 9 – 8 + 0
	~~1~~ 1 1 2 7	thousands column 9 – 1 + 3
		ten thousands column bring down the carried one

Cross out the one in the leftmost column. The final answer is 1127.

Original Problem multi-digit	New Problem with more space	
	^{1}	
1 5 3 1	1 5 3 1	ones column 10 – 5 + 1
<u>- 1 4 5</u>	9 9 9 ^{1}0	tens column 9 – 4 + 3
	<u>- 1 4 5</u>	hundreds column 9 – 1 + 5
	~~1~~ 1 3 8 6	thousands column 9 – 0 + 2
		ten thousands column bring down the carried one

Cross out the one in the leftmost column. The final answer is 1386.

Multiplication

Egyptian Multiplication

Our insight into the arithmetic of the ancient Egyptians comes primarily from the Rhind Papyrus. (Burton, p. 32) Written about 1650 B.C., it contains numerous multiplication and division problems. A cumbersome numeration system necessitated performing multiplication through repeated additions. This method for multiplication reduces the process to doubling and adding. It is a type of Duplation. Beginning with 1; make a chart, like the one below, by doubling the number in the left column each time. When two times the number in the left column is greater than the first factor, STOP!! Find the numbers in the left column that add up to the first factor. Add the corresponding numbers in the right column. This method does not translate well to other bases for beginning students.

Original Problem: 26 X 42

1	X	42	=	42
2	X	42	=	84
4	X	42	=	168
8	X	42	=	336
16	X	42	=	672

Since 2 x 16 = 32 which is greater than 26, STOP!!

26 = 16 + 8 + 2 so add

$$\begin{array}{r} 672 \\ 336 \\ +\ \underline{\ \ 64} \\ 1092 \end{array}$$

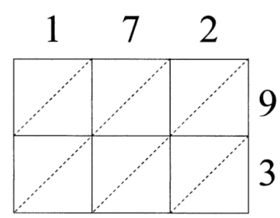
Therefore, the product of 26 and 42 is 1092, by Egyptian Multiplication.

Lattice Multiplication

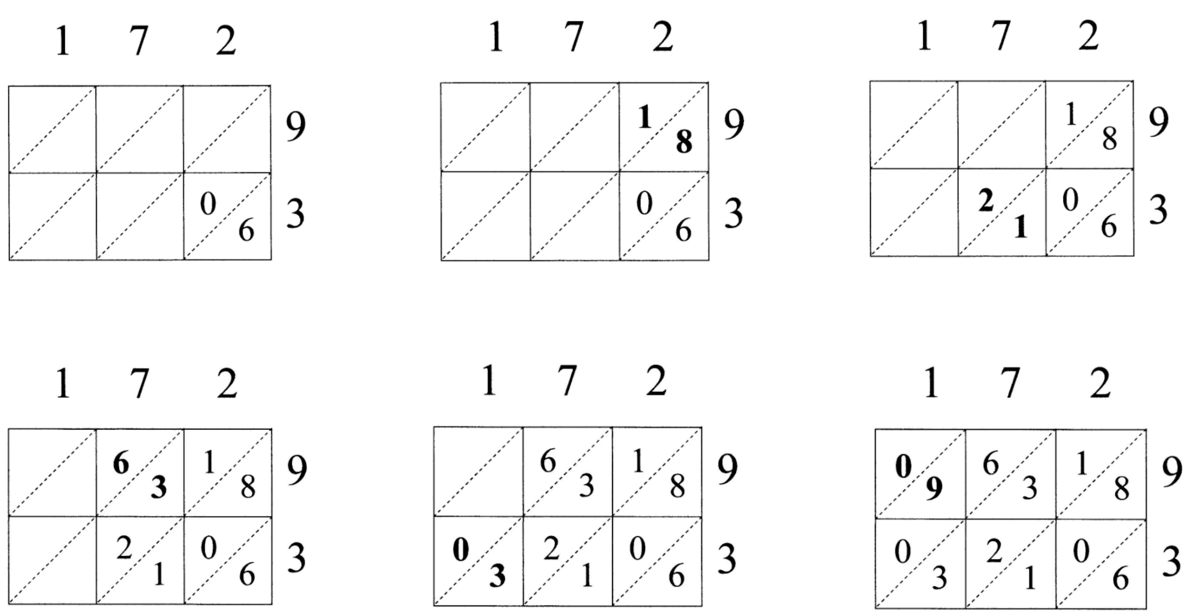
Lattice multiplication was introduced into Europe in the 13th century by the mathematician Leonardo of Pisa, more commonly known as Fibonacci. The lattice multiplication algorithm appeared in the very first printed arithmetic book, in Treviso, Italy in 1478. (Sonnabend, p. 170)

Let's look at the procedure using 172 × 93 as an example.

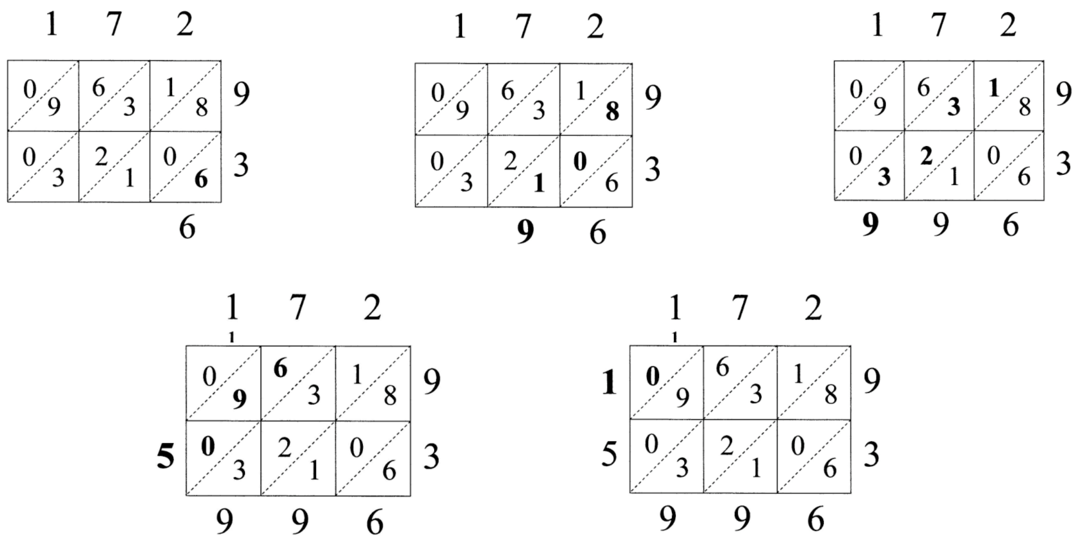
1) Start with a rectangular grid having the same number of columns (3) as the number of digits in one factor (172), and the same number of rows (2) as the number of digits in the other factor (93). Place the digits of one factor across the top and the digits from the other factor down the right side. Then draw diagonals from upper right to lower left to create the "lattice" design.



2) Multiply the corresponding digits and enter their products into the lattice as shown below, the tens digit going in the upper triangular region and the ones digit in the lower.

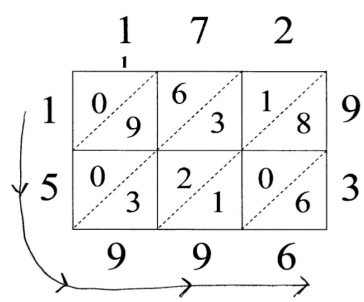


3) Finally, add along the diagonals, starting in the lower right corner, moving left, and carrying to the next diagonal when necessary.



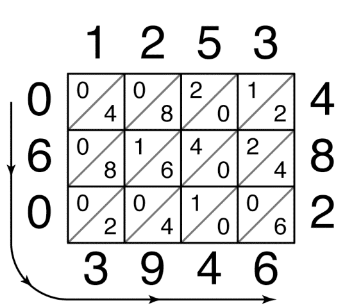
The answer is read, following the arrow, starting at the upper left.

$172 \times 93 = 15{,}996$

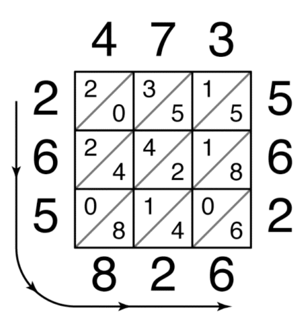


Two other examples:

$1253 \times 482 = 603{,}946$



$473 \times 562 = 265{,}826$



The advantage of this method is that all the multiplying is done first, then the adding. It is an excellent way to organize the multiplication of numbers with 3 or more digits each. It also translates wonderfully to non-decimal bases.

Russian Peasant Multiplication

The so-called "Russian Peasant" method, frequently seen in textbooks for pre-service teachers (Long, p. 208), is the ancient Egyptian "duplation and mediation" procedure. It has been renamed because of its use by Russian peasants, even into the 20th century. (Burton, p. 36) The advantage of this method is that the only multiplication facts we need to know are how to double and how to take half of a number.

Consider the following problem: 28 X 43

Take half of one factor (disregard remainders) and double the other factor.

	Take half		Double	
	28	X	43	
Halve 28 →	14		86	← Double 43
	7		172	
	3		344	
	1		688	

When we reach 1 in the halves column, we're through.

To get the answer, add the numbers in the doubles column that are opposite ODD numbers in the halves column:

172 + 344 + 688 = 1204

While this might be a tedious method to have as our only way to multiply, there are certain problems for which it is quite nice.

Take half		Double
32	X	25
16		50
8		100

Even though we're not technically finished, we know the answer!

Take half		Double
16	X	42
8		84
4		168
2		336
1		672

Since the only odd number in the halves column is 1, the last number in the doubles column is the answer!

Division

Egyptian Division

This method for division reduces the process to doubling and subtracting. It is the inverse of Egyptian multiplication. The divisor is doubled until the dividend is reached and the quotient is the missing factor. Beginning with 1, make a chart, like the one below, of doubles of the divisor. When the number in the right column exceeds the dividend, STOP!! Find the numbers in the right column that add up to the dividend. When the remaining number that you need to add is strictly less than the divisor, record it as the remainder. Add the corresponding numbers in the left column to produce the quotient. This method does not translate well to other bases for beginning students.

Original Problem: 352 divided by 9 or $9\overline{)352}$

1	X	9	=	9
2	X	9	=	18
4	X	9	=	36
8	X	9	=	72
16	X	9	=	144
32	X	9	=	288
64	X	9	=	576

Since 64 x 9 = 576, which is greater than 352, STOP!!
351 = 288 + 36 + 18 + 9. Add the number of 9's.

$$\begin{array}{r} 32 \\ 4 \\ 2 \\ +\ \ 1 \\ \hline 39 \end{array}$$

Thus, 39 X 9 = 351 so there is a remainder of 1. Therefore, when 352 is divided by 9, the quotient is 39 and the remainder is 1, by Egyptian Division.

Checking the Computational Form Yields:

288	=	32 X 9
+ 36	=	+ 4 X 9
+ 18	=	+ 2 X 9
+ 9	=	+ 1 X 9
351		39 Total

But we need 352 so we must add 1 to the 351. The quotient is 39 and the remainder is 1.

Verifying the Results Using a Different Computational Form:

$9\overline{)352}$	
− 288	32 X 9
64	
− 36	4 X 9
28	
− 18	2 X 9
10	
− 9	1 X 9
1	

Since 1 is strictly less than 9 it is the remainder.

Add 32 + 4 + 2 + 1 = 39 which is the quotient and the remainder is 1.

Therefore the quotient of 352 divided by 9 is 39 plus a remainder of 1, determined by Egyptian Division and verified by computational forms.

Repeated Subtraction (Scaffolding) Division Algorithm

Consider the problem: `136 ÷ 6`

We are literally asking ourselves the question
"How many times can 6 be subtracted from 136?

```
6 ⌐136 |
   -60 | 10  <----  We have subtracted
   ---                6 ten times
    76 |
   -60 | 10
   ---
    16 |
    -6 |  1
   ---
    10 |
    -6 |  1
   ---   ---
     4 | 22  <----  We have subtracted
                      6 a total of 22 times
```

The quotient is 22 and the remainder is 4. $136 = 6(22) + 4$

Another example: `496 ÷ 16`

```
16   ⌐496 |
     -160 | 10
      ---
      336 |
     -160 | 10
      ---
      176 |
     -160 | 10
      ---
       16 |
      -16 |  1
      --------
Remainder  ---->   0 | 31  <----   Quotient
```

The quotient is 31 and the remainder is 0 or using the standard form:
$496 = 31(16) + 0$

Many students are clever enough to construct their own algorithms for the basic operations of addition, subtraction, multiplication, and division. It is, of course, the responsibility of the teacher to make sure that these algorithms will work consistently. For those students who have trouble learning the standard algorithms or who just want to try some different ones for variety, we recommend these as a way to engage the students in the process of learning mathematics in an active fun environment. As is always true with mathematical or real life problem solving, showing a variety of methods yields creative adept learners who are willing to approach any challenge fearlessly.

References:

Bennett, Albert B. and Nelson, L. Ted (2004). *Mathematics for Elementary Teachers: A Conceptual Approach.* New York: McGraw-Hill.

Burton, David M. (2003). *The History of Mathematics*, 5th Edition. New York: McGraw-Hill.

Long, Calvin T. and DeTemple, Duane W. (2000). *Mathematical Reasoning for Elementary Teachers,* 2nd Edition. Reading, Massachusetts. Addison Wesley Longman.

Sonnabend, Thomas. (2004). *Mathematics for Teachers,* 3rd Edition. Belmont, California. Brooks Cole.

Webster's Seventh New Collegiate Dictionary. (1965). Springfield, Massachusetts. G. & C. Merriam Company.